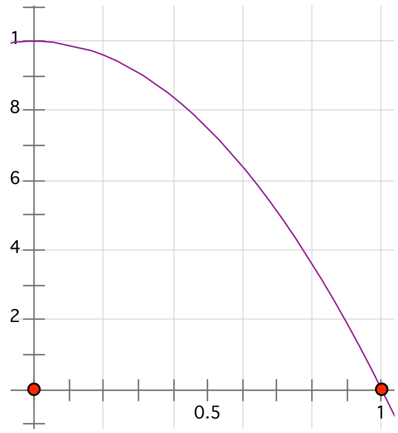
Approximating Area under a curve with rectangles

To find the area under a curve we approximate the area using rectangles and then use limits to find the area.

Example 1 Suppose we want to estimate $A =$ the area under the curve $y = 1 - x^2$, $0 \le x \le 1$.



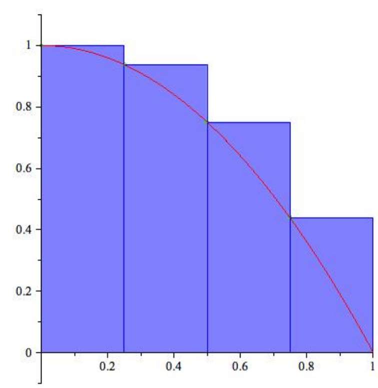
Left endpoint approximation To approximate the area under the curve, we can circumscribe the curve using rectangles as follows:

1. We divide the interval $[0,1]$ into 4 subintervals of equal length, $\Delta x = \frac{1-0}{4} = 1/4$. This divides the interval $[0,1]$ into 4 subintervals

$$[0,1/4],\ [1/4,1/2],\ [1/2,3/4],[3/4,1]$$

each with length $\Delta x = 1/4$. We label the endpoints of these subintervals as

$$x_0 = 0, \quad x_1 = 1/4, \quad x_2 = 2/4, \quad x_3 = 3/4, \quad x_4 = 1.$$



2. Above each subinterval draw a rectangle with height equal to the height of the function at the left end point of the subinterval. The values of the function at the endpoints of the subintervals are

x_i	$x_0 = 0$	$x_1 = 1/4$	$x_2 = 1/2$	$x_3 = 3/4$	$x_4 = 1$
$f(x_i) = 1 - x_i^2$	1	15/16	3/4	7/16	0

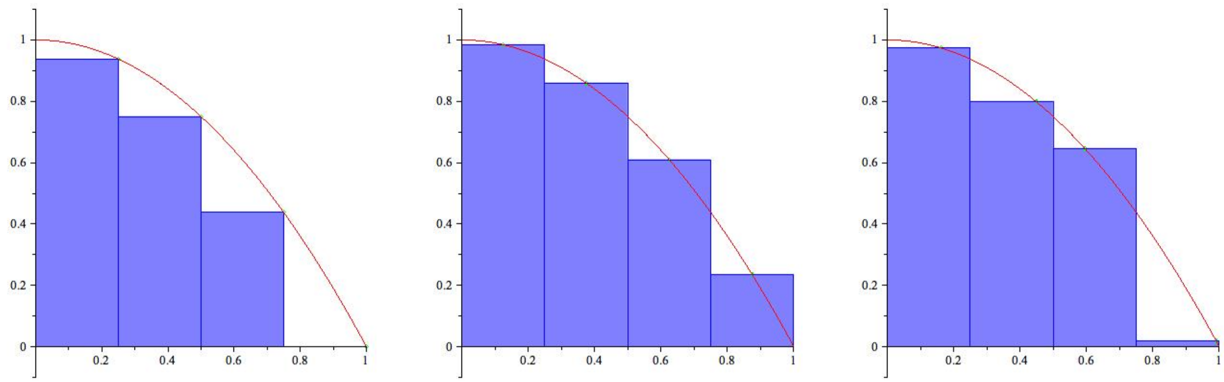
3. We use the sum of the areas of the approximating rectangles to approximate the area under the curve. We get

$$A \approx L_4 = \sum_{i=1}^{4} f(x_{i-1})\Delta x = f(x_0)\cdot\Delta x + f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x =$$

$$1 \cdot 1/4 + 15/16 \cdot 1/4 + 3/4 \cdot 1/4 + 7/16 \cdot 1/4 = 25/32 = 0.78125$$

L_4 is called the left endpoint approximation or the approximation using left endpoints (*of the subintervals*) and 4 approximating rectangles. We see in this case that $L_4 = 0.78125 > A$ (because the function is decreasing on the interval).

There is no reason why we should use the left end points of the subintervals to define the heights of the approximating rectangles, it is equally reasonable to use the right end points of the subintervals, or the midpoints or in fact a random point in each subinterval.



Right endpoint approximation In the picture on the left above, we use the right end point to define the height of the approximating rectangle above each subinterval, giving the height of the rectangle above $[x_{i-1}, x_i]$ as $f(x_i)$. This gives us inscribed rectangles. The sum of their areas gives us The right endpoint approximation, R_4 or the approximation using 4 approximating rectangles and right endpoints. Use the table above to complete the calculation:

$$A \approx R_4 = \sum_{i=1}^{4} f(x_i)\Delta x = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + f(x_4) \cdot \Delta x =$$

Is R_4 less than A or greater than A.

Midpoint Approximation In the picture in the center above, we use the midpoint of the intervals to define the height of the approximating rectangle. This gives us The Midpoint Approximation or The Midpoint Rule:

x_i^m = midpoint	$x_1^m = 0.125 = 1/8$	$x_2^m = 0.375 = 3/8$	$x_3^m = 0.625 = 5/8$	$x_4^m = 0.625 = 7/8$
$f(x_i^m) = 1 - (x_i^m)^2$	$63/64$	$55/64$	$39/64$	$15/64$

$$A \approx M_4 = \sum_{i=1}^{4} f(x_i^m)\Delta x = f(x_1^m)\Delta x + f(x_2^m)\Delta x + f(x_3^m)\Delta x + f(x_4^m)\Delta x =$$

General Riemann Sum We can use any point in the interval $x_i^* \in [x_{i-1}, x_i]$ to define the height of the corresponding approximating rectangle as $f(x_i^*)$. In the picture on the right we use random points

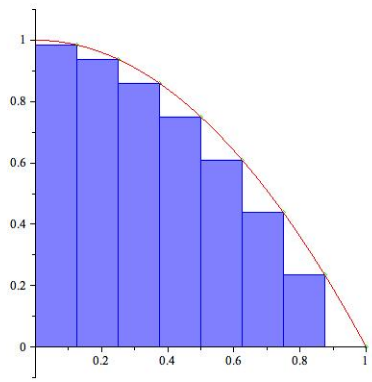
x_i^* in each interval to produce a rectangle with height $f(x_i^*)$. The sum of the areas of the rectangles gives us an approximation for A.

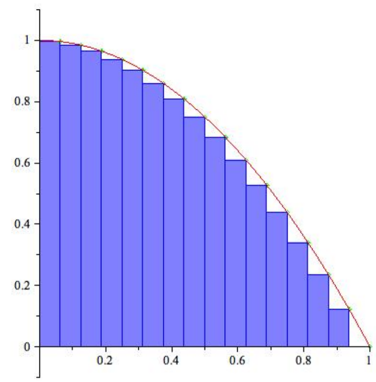
$$A \approx \sum_{i=1}^{4} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$$

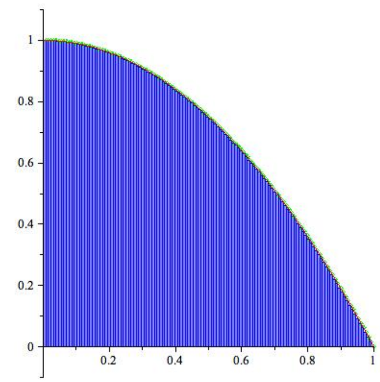
Increasing The Number of Rectangles and taking Limits

When we increase the number of rectangles (of equal width) used, using a smaller value for Δx (= the width of the rectangles), we get a better approximation to the area. You can see this in the pictures below for A = the area under the curve $y = 1 - x^2$, $0 \le x \le 1$. The pictures show the right end point approximations to A with $\Delta x = 1/8, 1/16$ and $1/128$ respectively:

$R_8 = .6015625000,$ $R_{16} = .6347656250,$ $R_{128} = .6627502441$

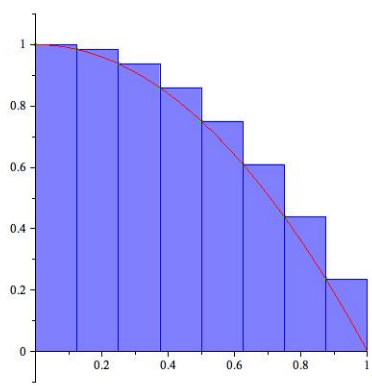


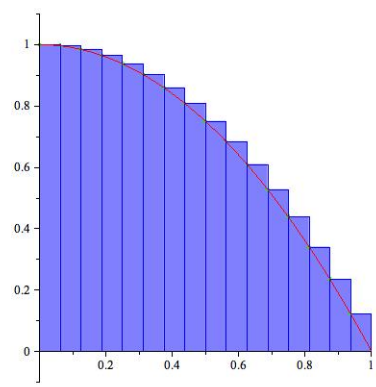


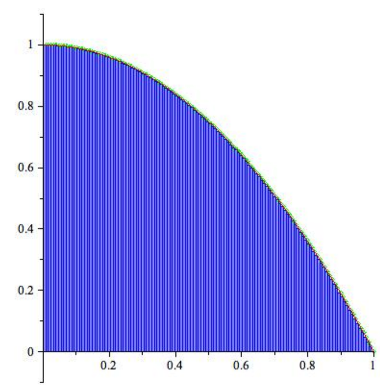


The pictures below show the left end point approximations to the area, A, with $\Delta x = 1/8, 1/16$ and $1/128$ respectively.

$L_8 = .7265625000,$ $L_{16} = .6972656250,$ $L_{128} = .6705627441$







We see that the room for error decreases as the number of subintervals increases or as $\Delta x \to 0$. Thus we see that

$$\boxed{A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

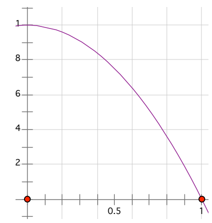
where x_i^* is any point in the interval $[x_{i-1}, x_i]$. In fact this is our definition of the area under the curve on the given interval. (If A denotes the area beneath $f(x) \ge 0$ where f is continuous on the interval $[a, b]$, then each interval $[x_{i-1}, x_i]$ has length $\Delta x = \frac{b-a}{n}$ for any given n.)

Calculating Limits of Riemann sums

The following formulas are sometimes useful in calculating Riemann sums. I have attached some visual proofs at the end of the lecture.

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{(2n+1)n(n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$$

Let us now consider **Example 1**. We want to find $A =$ the area under the curve $y = 1 - x^2$ on the interval $[a, b] = [0, 1]$.



We know that $A = \lim_{n\to\infty} R_n$, where R_n is the right endpoint approximation using n approximating rectangles.

We must calculate R_n and than find $\lim_{n\to\infty} R_n$.

1. We divide the interval $[0, 1]$ into n strips of equal length $\Delta x = \frac{1-0}{n} = 1/n$. This gives us a partition of the interval $[0, 1]$,

$$x_0 = 0, \quad x_1 = 0 + \Delta x = 1/n, \quad x_2 = 0 + 2\Delta x = 2/n, \quad \ldots, \quad x_{n-1} = (n-1)/n, \quad x_n = 1.$$

2. We will use the right endpoint approximation R_n.

3. The heights of the rectangles can be found from the table below:

x_i	$x_0 = 0$	$x_1 = 1/n$	$x_2 = 2/n$	$x_3 = 3/n$	$\ldots$	$x_n = n/n$
$f(x_i) = 1 - (x_i)^2$	1	$1 - 1/n^2$	$1 - 2^2/n^2$	$1 - 3^2/n^2$	$\ldots$	$1 - n^2/n^2$

4.

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x =$$

$$\left(1 - \frac{1}{n^2}\right)\frac{1}{n} + \left(1 - \frac{2^2}{n^2}\right)\frac{1}{n} + \left(1 - \frac{3^2}{n^2}\right)\frac{1}{n} + \cdots + \left(1 - \frac{n^2}{n^2}\right)\frac{1}{n} =$$

$$\frac{1}{n} - \frac{1}{n^2}\left(\frac{1}{n}\right) + \frac{1}{n} - \frac{2^2}{n^2}\left(\frac{1}{n}\right) + \frac{1}{n} - \frac{3^2}{n^2}\left(\frac{1}{n}\right) + \cdots + \frac{1}{n} - \frac{n^2}{n^2}\left(\frac{1}{n}\right) =$$

5. Finish the calculation above and find $A = \lim_{n\to\infty} R_n$ using the formula for the sum of squares and calculating the limit as if R_n were a rational function with variable n.

Also $\quad A = \lim_{n\to\infty} L_n$

From Part 3, we have $\Delta x = 1/n$ and

$$L_n = \frac{1}{n} + \Big(1 - \frac{1}{n^2}\Big)\frac{1}{n} + \Big(1 - \frac{2^2}{n^2}\Big)\frac{1}{n} + \Big(1 - \frac{3^2}{n^2}\Big)\frac{1}{n} + \cdots + \Big(1 - \frac{(n-1)^2}{n^2}\Big)\frac{1}{n}$$

$$\frac{1}{n} + \frac{1}{n} - \frac{1}{n^2}\Big(\frac{1}{n}\Big) + \frac{1}{n} - \frac{2^2}{n^2}\Big(\frac{1}{n}\Big) + \frac{1}{n} - \frac{3^2}{n^2}\Big(\frac{1}{n}\Big) + \cdots + \frac{1}{n} - \frac{(n-1)^2}{n^2}\Big(\frac{1}{n}\Big) =$$

grouping the $\frac{1}{n}$'s together, we get

$$= \frac{n}{n} - \frac{1}{n}\Big[\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\Big]$$

$$= 1 - \frac{1}{n^3}\big[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2\big]$$

$$= 1 - \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = 1 - \frac{1}{n^3}\Big[\frac{(2(n-1)+1)(n-1)((n-1)+1)}{6}\Big]$$

$$= 1 - \frac{1}{n^3}\Big[\frac{(2n-1)(n-1)(n)}{6}\Big]$$

$$= 1 - \frac{n}{6n^3}(2n-1)(n-1)$$

$$= 1 - \frac{2n + \text{ smaller powers of } n}{6n^2}$$

So

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty}\Big[1 - \frac{2n + \text{ smaller powers of } n}{6n^2}\Big] = 1 - \frac{2}{6} = 2/3.$$

Riemann Sums in Action: Distance from Velocity/Speed Data

To estimate distance travelled or displacement of an object moving in a straight line over a period of time, from discrete data on the velocity of the object, we use a Riemann Sum. If we have a table of values:

time $= t_i$	$t_0 = 0$	t_1	t_2	$\ldots$	t_n
velocity $= v(t_i)$	$v(t_0)$	$v(t_1)$	$v(t_2)$	$\ldots$	$v(t_n)$

where $\Delta t = t_i - t_{i-1}$, then we can approximate the displacement on the interval $[t_{i-1}, t_i]$ by $v(t_{i-1}) \times \Delta t$ or $v(t_i) \times \Delta t$. Therefore the total displacement of the object over the time interval $[0, t_n]$ can be approximated by

$$\boxed{\text{Displacement} \approx v(t_0)\Delta t + v(t_1)\Delta t + \cdots + v(t_{n-1})\Delta t \quad \text{Left endpoint approximation}}$$

or

$$\boxed{\text{Displacement} \approx v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t \quad \text{Right endpoint approximation}}$$

These are obviously Riemann sums related to the function $v(t)$, hinting that there is a connection between the area under a curve (such as velocity) and its antiderivative (displacement). This is indeed the case as we will see later.

When we use speed $= |\text{velocity}|$ instead of velocity. the above formulas translate to

$$\boxed{\text{Distance Travelled} \approx |v(t_0)|\Delta t + |v(t_1)|\Delta t + \cdots + |v(t_{n-1})|\Delta t}$$

and

$$\boxed{\text{Distance Travelled} \approx |v(t_1)|\Delta t + |v(t_2)|\Delta t + \cdots + |v(t_n)|\Delta t}$$

Example The following data shows the speed of a particle every 5 seconds over a period of 30 seconds. Give the left endpoint estimate for the distance travelled by the particle over the 30 second period.

time in s $= t_i$	0	5	10	15	20	25	30
velocity in m/s $= v(t_i)$	50	60	65	62	60	55	50

$$L = |v(t_0)|\Delta t + |v(t_1)\Delta t + \cdots + |v(t_6)|\Delta t$$

$$= 50(5) + 60(5) + 65(5) + 62(5) + 60(5) + 55(5)$$

$$= 5[50 + 60 + 65 + 62 + 60 + 55] = 1760m.$$

The above sum is a Riemann sum, telling us that the distance travelled is approximately the area under the (absolute vale of velocity) curve.... .... hmmmmm intetresting........ remember speed $= |v(t)| =$ derivative of distance travelled. ....

Appendix

Area under a curve, Summary of method using Riemann sums.

To find the area under the curve $y = f(x)$ on the interval $[a, b]$, where $f(x) \geq 0$ for all x in $[a, b]$ and continuous on the interval:

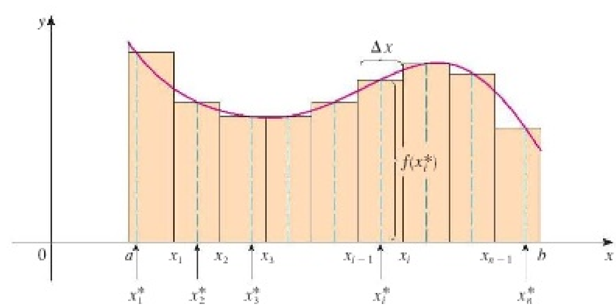
1. Divide the interval into n strips of equal width $\Delta x = \frac{b-a}{n}$. This divides the interval $[a, b]$ into n subintervals:
$$[x_0 = a, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, [x_{n-1}, x_n = b].$$
(Note $x_1 = a+\Delta x, \;\; x_2 = a+2\Delta x, \;\; x_3 = a+3\Delta x, \;\; \ldots \;\; x_{n-1} = a+(n-1)\Delta x, \;\; x_n = a+n\Delta x = b.$)

2. For each interval, pick a sample point, x_i^* in the interval $[x_{i-1}, x_i]$.

3. Construct an approximating rectangle above the subinterval $[x_{i-1}, x_i]$ with height $f(x_i^*)$. The area of this rectangle is $f(x_i^*)\Delta x$.

4. The total area of the approximating rectangles is
$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$
(The sum is called a Riemann Sum.)

5. We define the area under the curve $y = f(x)$ on the interval $[a, b]$ as
$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$
$$= \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$
$$= \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x].$$

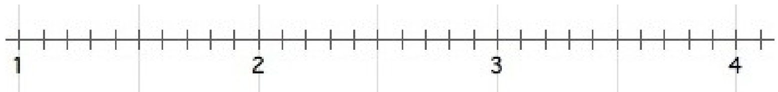


(In a more advanced course, you would prove that all of these limits give the same number A which we use as a measure/definition of the area under the curve)

Extra Example Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 4$ using six approximating rectangles and

$\Delta x = \frac{b-a}{n} =$____________ , where $[a, b] = [1, 4]$ and $n = 6$.

Mark the points $x_0, x_1, x_2, \ldots, x_6$ which divide the interval $[1, 4]$ into six subintervals of equal length on the following axis:



Fill in the following tables:

x_i	$x_0 =$	$x_1 =$	$x_2 =$	$x_3 =$	$x_4 =$	$x_5 =$	$x_6 =$
$f(x_i) = 1/x_i$							

(a) Find the corresponding right endpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$R_6 =$

(b) Find the corresponding left endpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$L_6 =$

(c) Fill in the values of $f(x)$ at the midpoints of the subintervals below:

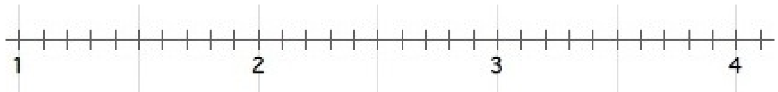
midpoint $= x_i^m$	$x_1^m =$	$x_2^m =$	$x_3^m =$	$x_4^m =$	$x_5^m =$	$x_6^m =$
$f(x_i^m) = 1/x_i^m$						

Find the corresponding midpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$M_6 =$

Extra Example Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 4$ using six approximating rectangles and

$\Delta x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{1}{2}$, where $[a, b] = [1, 4]$ and $n = 6$.
Mark the points $x_0, x_1, x_2, \ldots, x_6$ which divide the interval $[1, 4]$ into six subintervals of equal length on the following axis:



Fill in the following tables:

x_i	$x_0 = 1$	$x_1 = 3/2$	$x_2 = 2$	$x_3 = 5/2$	$x_4 = 3$	$x_5 = 7/2$	$x_6 = 4$
$f(x_i) = 1/x_i$	1	2/3	1/2	2/5	1/5	2/7	1/4

(a) Find the corresponding right endpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$$\begin{aligned} R_6 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{7} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \\ &= \frac{2}{6} + \frac{1}{4} + \frac{2}{10} + \frac{1}{6} + \frac{2}{14} + \frac{1}{8} = 1.217857 \end{aligned}$$

(b) Find the corresponding left endpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$$\begin{aligned} L_6 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= 1 \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{7} \cdot \frac{1}{2} \\ &= \frac{1}{2} + \frac{2}{6} + \frac{1}{4} + \frac{2}{10} + \frac{1}{6} + \frac{2}{14} = 1.59285 \end{aligned}$$

(c) Fill in the values of $f(x)$ at the midpoints of the subintervals below:

midpoint $= x_i^m$	$x_1^m = 5/4$	$x_2^m = 7/4$	$x_3^m = 9/4$	$x_4^m = 11/4$	$x_5^m = 13/4$	$x_6^m = 15/4$
$f(x_i^m) = 1/x_i^m$	4/5	4/7	4/9	4/11	4/13	4/15

Find the corresponding midpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$$M_6 = \sum_{i=1}^{6} f(x_i^*)\Delta x$$

$$= \frac{4}{5} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{1}{2} + \frac{4}{9} \cdot \frac{1}{2} + \frac{4}{11} \cdot \frac{1}{2} + \frac{4}{13} \cdot \frac{1}{2} + \frac{4}{15} \cdot \frac{1}{2} = 1.376934$$

Extra Example Find the area under the curve $y = x^3$ on the interval $[0, 1]$.
We know that $A = \lim_{n\to\infty} R_n$, where R_n is the right endpoint approximation using n approximating rectangles.

We must calculate R_n and than find $\lim_{n\to\infty} R_n$.

1. We divide the interval $[0, 1]$ into n strips of equal length $\Delta x = \frac{1-0}{n} = 1/n$. This gives us a partition of the interval $[0, 1]$,

$$x_0 = 0, \quad x_1 = 0 + \Delta x = 1/n, \quad x_2 = 0 + 2\Delta x = 2/n, \quad \dots, \quad x_{n-1} = (n-1)/n, \quad x_n = 1.$$

2. We will use the right endpoint approximation R_n.

3. The heights of the rectangles can be found from the table below:

x_i	$x_0 = 0$	$x_1 = 1/n$	$x_2 = 2/n$	$x_3 = 3/n$	$\dots$	$x_n = n/n$
$f(x_i) = (x_i)^3$	0	$1/n^3$	$2^3/n^3$	$3^3/n^3$	$\dots$	n^3/n^3

4.
$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x =$$
$$\left(\frac{1}{n^3}\right)\frac{1}{n} + \left(\frac{2^3}{n^3}\right)\frac{1}{n} + \left(\frac{3^3}{n^3}\right)\frac{1}{n} + \cdots + \left(\frac{n^3}{n^3}\right)\frac{1}{n} =$$
$$= \sum_{i=1}^{n} \frac{i^3}{n^4} = \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

5.
$$A = \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4} = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2}$$
$$= \lim_{n\to\infty} \frac{1}{4} \cdot \frac{(n+1)}{n} \cdot \frac{(n+1)}{n} = \frac{1}{4}.$$

Extra Example, estimates from data on rate of change The same principle applies to estimating Volume from discrete data on its rate of change:
Oil is leaking from a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

time in h	0	1	2	3	4	5	6	7	8
leakage in gal/h	50	70	97	136	190	265	369	516	720

The following gives the right endpoint estimate of the amount of oil that has escaped from the tanker after 8 hours:

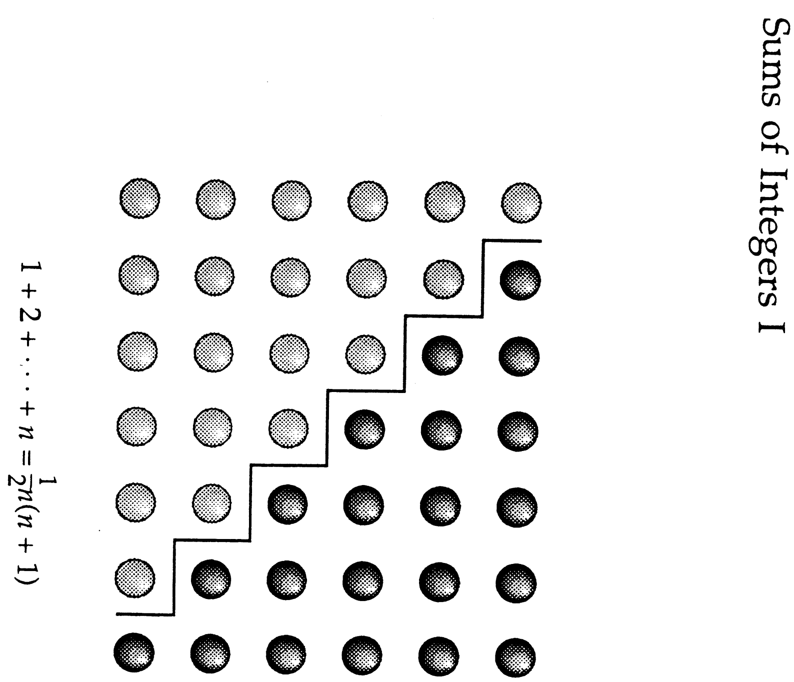
$$R_8 = 70 \cdot 1 + 97 \cdot 1 + 136 \cdot 1 + 190 \cdot 1 + 265 \cdot 1 + 369 \cdot 1 + 516 \cdot 1 + 720 \cdot 1 = 2363 \text{ gallons.}$$

The following gives the right endpoint estimate of the amount of oil that has escaped from the tanker after 8 hours:

$$L_8 = 50 \cdot 1 + 70 \cdot 1 + 97 \cdot 1 + 136 \cdot 1 + 190 \cdot 1 + 265 \cdot 1 + 369 \cdot 1 + 516 \cdot 1 = 1693 \text{ gallons.}$$

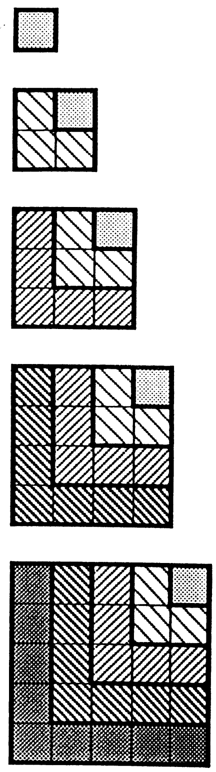
Since the flow of oil seems to be increasing over time, we would expect that
$L_8 <$ true volume leaked $< R_8$ or the true volume leaked in the first 8 hours is somewhere between 1693 and 2363 gallons.

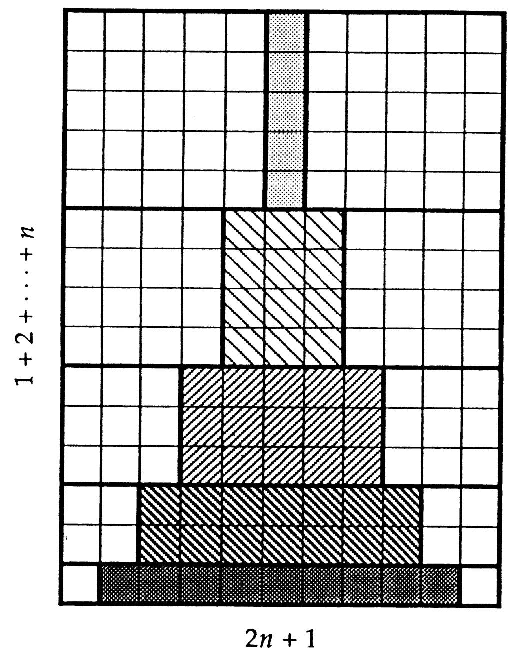
Visual proof of formula for the sum of integers:



Sums of Squares II

$$3(1^2 + 2^2 + \cdots + n^2) = (2n + 1)(1 + 2 + \cdots + n)$$





—Martin Gardner and Dan Kalman
(independently)